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ON PARTICULAR SOLUTIONS OF THE PROBLEM OF MOTION OF A GYROSCOPE IN GIMBAL MOUNT

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Particular solutions of equations of motion of a heavy gyroscope in gimbal mount with the outer gimbal axis horizontal was considered in [1, 2].

Particular solutions of this problem are considered below in the case, when the axis of rotation of the outer gimbal is horizontal and the center of gravity of the gyroscope and of the casing are not lying on the axis of symmetry of the gyroscope ellipsoid of inertia but in a plane passing through the axis of symmetry perpendicularly to the axis of rotation of the inner gimbal. The latter solutions also complement each other symmetrically, similarly to those mentioned above.

The fixed system of coordinates $O\xi\eta\zeta$ is permanently attached to the axis of rotation of the outer gimbal (Fig. 1). The ξ-axis lies on the axis of rotation of that gimbal. The system of coordinate axes $Ox_2y_2z_2$ is permanently attached to the outer gimbal. The x_2- and y_2-axes coincide with the axes of rotation of the outer and inner gimbals, respectively. The system of coordinates $Ox_1y_1z_1$ is permanently attached to the casing. The y_1-axis is directed along tha casing axis of rotation and the z_1-axis along the rotor spin axis. Axes x_1, y_1 and z_1 are the principal axes of the ellipsoid of the casing inertia about the fixed point O.

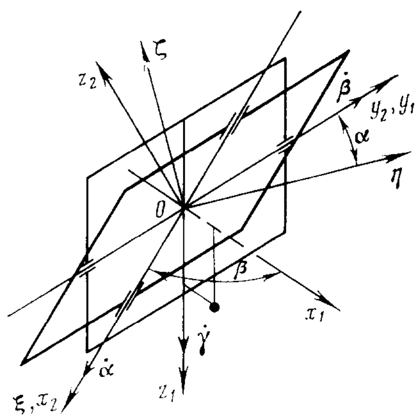

Fig. 1

Let us assume that the ellipsoid of rotor inertia about point O is the ellipsoid of rotation of the casing about the z_1-axis.

We use the following notation: α is the angle of turn of the outer gimbal; β is the angle of turn of the casing (inner gimbal), γ is the angle of turn of the rotor in the casing (angle of spin of the gyroscope about the z_1-axis); A_2 is the moment of inertia of the outer gimbal about the ξ-axis; A_1, B_1

and C_1 are the principal moments of inertia of the casing about the x_1-, y_1- and z_1-axes; A, $B = A$, C are the moments of inertia of the rotor about the same axes.

The table of cosines of angles between the axes of the system of coordinates ξ, η, ζ and x_1, y_1, z_1 is of the form

$$\begin{array}{cccc} & x_1 & y_1 & z_1 \\ \xi & \cos\beta & 0 & \sin\beta \\ \eta & \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ \zeta & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta \end{array} \tag{1}$$

The overall kinetic energy of the system is

$$2T = (A_0 - C_0 \sin^2\beta)\,\alpha'^2 + B_0\beta'^2 + C(\gamma' + \alpha'\sin\beta)^2$$
$$A_0 = A + A_1 + A_2, \qquad B_0 = A + B_1, \qquad C_0 = A + A_1 - C_1$$

We assume the absence of friction in bearings and that the force of gravity acts on the system.

Let the distribution of masses in the considered system be such that the conditions $A_0 = B_0$ and $C_0 = 0$ are satisfied. We express the equations of motion of the system in the form of Lagrange equations in variables α, β and γ

$$\begin{aligned} &d\,[A_0\alpha' + C\sin\beta\,(\gamma' + \alpha'\sin\beta)]\,/\,dt = \partial U\,/\,\partial\alpha \\ &A_0\beta'' - C\cos\beta\,(\gamma' + \alpha'\sin\beta)\,\alpha' = \partial U\,/\,\partial\beta \\ &d\,\,[C(\gamma' + \alpha'\sin\beta)]/dt = \partial U\,/\,\partial\gamma \end{aligned} \tag{2}$$

For the kinetic energy we have the integral

$$A_0(\alpha'^2 + \beta'^2) + C(\gamma' + \alpha'\sin\beta)^2 = 2U + 2h \tag{3}$$

We direct the ζ-axis of the fixed system of coordinates vertically upward and the ξ-axis horizontally (Fig. 1), and assume that the center of gravity of the rotor and casing lies in the x_1z_1-plane, where its coordinates are x_0 and z_0. Then, in accordance with Table (1), the force function is of the form

$$\begin{aligned} &U = mg\cos\alpha\,(x_0\sin\beta - z_0\cos\beta) \\ &(m = \text{mass of rotor and casing}) \end{aligned} \tag{4}$$

In this case the equations of motion (2) admit besides integral (3) the integral which corresponds to the ignorable coordinate γ

$$\gamma' + \alpha'\sin\beta = r_0 \qquad (r_0 = \text{const}) \tag{5}$$

For the particular value $r_0 = 0$ we have one more first integral

$$A_0\alpha'\beta' = mg\sin\alpha\,(x_0\cos\beta + z_0\sin\beta) + l \qquad (l = \text{const}) \tag{6}$$

We pass to variables

$$\sigma = \beta + \alpha, \qquad \omega = \beta - \alpha \tag{7}$$

The equations of motion for the force function (4) and $r_0 = 0$ yield the first integrals

$$\begin{aligned} &A_0\sigma'^2 = 2mg\,(x_0\sin\sigma - z_0\cos\sigma) + l_1 \\ &A_0\omega'^2 = 2mg\,\,(x_0\sin\omega - z_0\cos\omega) + l_2 \\ &(l_1 + l_2 = 4h,\; l_1 - l_2 = 4l) \end{aligned} \tag{8}$$

In accordance with integrals (3) and (6) constants h, l and l_1, l_2 are related by formulas appearing in parentheses. We introduce the following notation for the constants:

$$a=\frac{h+l}{3A_0}=\frac{l_1}{6A_0},\quad b=\frac{h-l}{3A_0}=\frac{l_2}{6A_0},\quad n=\frac{mgx_0}{A_0},\quad c=\frac{mgz_0}{A_0} \tag{9}$$

We substitute variables

$$n\sin\sigma-c\cos\sigma+a=-2\mu,\quad n\sin\omega-c\cos\omega+b=-2\nu \tag{10}$$

Equation (8) in new variables becomes

$$(dw/dt)^2=4w^3-(3\varkappa^2+\Delta^2)w-\varkappa(\varkappa^2-\Delta^2) \tag{11}$$
$$(w=\mu \text{ for } \varkappa=a,\ w=\nu \text{ for } \varkappa=b,\ \Delta^2=c^2+n^2)$$

If we set $d\tau/dt=\pm 1$, Eq. (11) is satisfied by the Weierstrass function $\wp_1(\tau)$ with invariants g_2' and g_3', and $\wp_2(\tau)$ with invariants g_2'' and g_3'' defined by

$$g_2'=3a^2+\Delta^2,\quad g_3'=a(a^2-\Delta^2),\quad g_2''=3b^2+\Delta^2,\quad g_3''=b(b^2-\Delta^2)$$

Let us take the plus sign. In accordance with (10) we obtain

$$-2\wp_1(\tau)=n\sin\sigma-c\cos\sigma+a,\ -2\wp_2(\tau)=n\sin\omega-c\cos\omega+b$$

The first equality yields for $\sin\sigma$ a quadratic equation whose solution is

$$\sin\sigma=1/\Delta^2\{-n(2\wp_1+a)\pm n[\Delta^2-(2\wp_1+a)^2]^{1/2}\}$$

We obtain a similar expression for $\sin\omega$.

Parameters α and β are expressed in terms of Weierstrass functions with the substitution of variables (7) and of the derived expressions for $\sin\sigma$ and $\sin\omega$. Angle γ is calculated by the quadrature for $r_0=0$.

The right-hand part of Eq. (11) is a third power polynomial in w, which has three real roots

$$e_{1,2}=1/2(\mp\Delta-\varkappa),\quad e_3=\varkappa$$

Let $n>0$ and $c>0$. We can then deduct from equality (10) the following conclusion.

Let us consider the z_1x_1-plane (Fig. 2) in which we draw orthogonal axes z^*x^* obtained by the rotation of the z_1- and x_1-axes by the angle $\pi/2-\lambda$ in the counterclockwise direction. The center of gravity of the rotor and casing, whose initial coordinates were z_0 and x_0 in the new coordinates, become

$$z^*=x_0\cos\lambda+z_0\sin\lambda \tag{12}$$
$$x^*=x_0\sin\lambda-z_0\cos\lambda$$

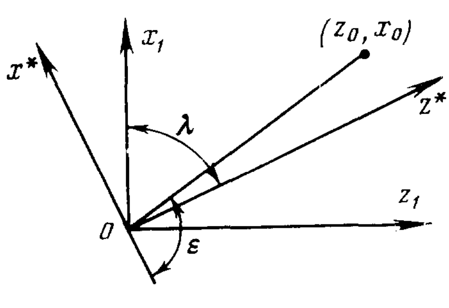

Fig. 2

We denote by ε the angle of inclination of the radius vector of point (z_0, x_0) to the x^*-axis. If $\lambda=\sigma$, then

$$\sin\varepsilon=\frac{x_0\sin\sigma-z_0\cos\sigma}{(z_0^2+x_0^2)^{1/2}}=-\frac{2\mu+a}{\Delta}$$

Hence the variable μ varies within the limits $e_1\leqslant\mu\leqslant e_2$. A similar relationship can be derived for the variable ν: $e_1\leqslant\nu\leqslant e_2$. Actual mechanical motions occur for $e_3>e_1$. This condition for variables μ and ν yields, respectively, the inequalities

$$h\pm l+mg(x_0^2+z_0^2)^{1/2}>0$$

For other restrictions on the position of the center of gravity of the rotor and casing, for instance, $n > 0$, $c < 0$ (etc.), it is possible to determine the restrictions on constants of integrals for actual mechanical motion. If the η-axis of the fixed system of coordinates is directed vertically upward, then in accordance with (1) the force function assumes the form

$$U = -mg \sin \alpha \, (x_0 \sin \beta - z_0 \cos \beta) \tag{13}$$

The equations of motion (2) admit first integrals of kinetic energy and γ corresponding to the ignorable coordinate. For the particular value $r_0 = 0$ there exists one more integral

$$A_0 \alpha' \beta' = mg \cos \alpha \, (x_0 \cos \beta + z_0 \sin \beta) + l$$

After the substitution of variables (7), the equations of motion yield first integrals

$$\begin{aligned} A_0 \sigma'^2 &= 2mg \, (x_0 \cos \sigma + z_0 \sin \sigma) + l_1 \\ A_0 \omega'^2 &= -2\, mg \, (x_0 \cos \omega + z_0 \sin \omega) + l_2 \end{aligned} \tag{14}$$

Euler's angles and restrictions on constants of integrals for actual mechanical motion are computed in the manner described above.

Indicated motions may be interpreted in terms of previously determined motions [1, 2] as follows.

The distance of the center of gravity from the coordinate origin in the $z_1 x_1$-plane (Fig. 2) is $A_0 \Delta \,/\, mg$

$$z^* = \frac{A_0}{mg} \Delta \sin \varepsilon, \qquad x^* = -\frac{A_0}{mg} \Delta \cos \varepsilon, \qquad \lambda' = \varepsilon' \tag{15}$$

Taking into account (9), (12) and (15), we can write Eqs. (8) as follows:

$$\begin{gathered} \varepsilon'^2 = 2\,(-\Delta \cos \varepsilon + 3\varkappa) \\ (\sigma = \lambda \;\text{ for }\; \varkappa = a, \;\; \omega = \lambda \;\text{ for }\; \varkappa = b) \end{gathered} \tag{16}$$

In a similar manner Eqs. (14) can be written as

$$\begin{gathered} \varepsilon'^2 = 2\,(j \Delta \sin \varepsilon + 3\varkappa) \\ (\sigma = \lambda \;\text{ for }\; \varkappa = a, \; j = +1; \;\; \omega = \lambda \;\text{ for }\; \varkappa = b, \; j = -1) \end{gathered} \tag{17}$$

If in Eqs. (16) we make the substitution

$$\begin{array}{cccccccccccccc} \psi & \theta & x_1 & y_1 & z_1 & x & y & z & I & A^\circ & B^\circ & C^\circ & \alpha & \beta \\ \alpha & \beta - \pi/2 & \eta & \zeta & \xi & y_1 & x_1 & -z_1 & A_2 & B_1 & A_1 & C_1 & \sigma - \pi/2 & \omega - \pi/2 \end{array}$$

they coincide with Eqs. (9) in [2], while the same substitution in Eqs. (17) yields Eqs. (9) in [1].

The investigation of motion in the $z^* x^*$-axes presents certain inconveniences, since these alter their position with time in relation to the fundamental gimbal elements (which is not the case with the $z_1 x_1$-axes) and can, apparently, lead to various interpretations of the motion.

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